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# Enhancing the relationship between ring-averaged resonance driving term and dynamic aperture through frequency diffusion

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## TECHNICAL REPORT

# Enhancing the relationship between ring-averaged resonance driving term and dynamic aperture through frequency diffusion

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**ABSTRACT.** In our previous work, it was found that reducing the variation of resonance driving terms (RDTs) along the longitudinal position of a storage ring, quantified by the ring-averaged RDT, effectively enlarges the area of dynamic aperture (DA). In addition to the area of DA, conventionally determined by locating the survival boundary through particle tracking, the stability of surviving particle motion is also important, since more stable motion implies a more robust DA. In this paper, by using frequency map analysis (FMA) to quantify the motion stability, the relationship between the ring-averaged RDT and DA is further enhanced. For FMA with grid tracking, we can obtain the number of surviving particles ( $N_{\text{particle}}$ ) and their frequency diffusion rates. It is found that  $N_{\text{particle}}$  exhibits a stronger correlation with ring-averaged RDT than the DA area obtained with the line search in elegant, which was used in our previous work. When frequency diffusion rates are taken into account, for nonlinear solutions with similar  $N_{\text{particle}}$ , those with smaller ring-averaged RDT on average exhibit lower frequency diffusion rates, indicating more stable particle motion. This becomes more apparent when amplitude-dependent tune shifts are constrained. While for nonlinear solutions with similar ring-averaged RDT, those with larger  $N_{\text{particle}}$  generally have higher frequency diffusion rates, and smaller  $N_{\text{particle}}$  lower rates. In addition, the optimization for reducing ring-averaged RDT is compared with the tracking-based optimization for lowering frequency diffusion rates. It is found that the two optimizations find almost the same solution region with larger DAs and lower diffusion rates.

**KEYWORDS:** Beam dynamics; Accelerator modelling and simulations (multi-particle dynamics, single-particle dynamics); Instrumentation for synchrotron radiation accelerators

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## 1 Introduction

The dynamic aperture (DA) of a storage ring affects both beam injection and elastic gas scattering lifetime, making the optimization of DA one of the primary tasks in the design of low-emittance storage ring lattices [1]. Apart from the size of DA, the stability of particle motion within DA is also crucial. There are many dynamic indicators to assess the stability of motion, such as Lyapunov exponent [2], frequency map analysis (FMA) [3–5] and other indicators in refs. [6, 7]. FMA is widely used to analyze the influence of resonances on DA. In FMA, in addition to the tunes of each particle, the change of tunes during particle motion, known as frequency diffusion, is also calculated. A lower frequency diffusion rate indicates more stable particle motion. In refs. [8–10], reducing the total frequency diffusion rate was used as an objective for DA optimization, resulting in more robust DAs.

Controlling nonlinear driving terms [11] is a widely applied analytical approach for designing lattice and optimizing DA [12–17]. In this approach, the nonlinear part of one-turn map is expanded using the resonance basis, yielding a series of nonlinear driving terms, including resonance driving terms (RDTs) and tune shift terms. Minimizing RDTs helps to suppress the corresponding resonances and thus enlarge DA, which requires much less computational cost than the tracking-based DA optimization. In our previous work [18], the theoretical analysis via the Baker-Campbell-Hausdorff formula reveals that reducing the variation of lower-order RDTs along the longitudinal position can control higher-order nonlinear terms and suppress higher-order resonances. Consequently, it was demonstrated that reducing the longitudinal variation of RDTs, quantified by the ring-averaged RDT, effectively increases DA area, which is much more effective than reducing the commonly-used one-turn RDTs. For a pair of thin sextupoles separated by an exact  $-I$  transformation, the nonlinear effects cancel completely. When the finite lengths of practical sextupoles are taken into account, higher-order nonlinear terms are present, and reducing the ring-averaged RDT is effective for a practical  $-I$  cancellation scheme [18]. Our previous work did not account for differences in particle motion stability within DA. Both expanding DA and improving particle motion stability reflect the suppression of nonlinear effects, thereby enhancing overall DA performance. To further analyze the influence of reducing ring-averaged RDT on DA, this paper investigates the relationship between ring-averaged RDT and DA using FMA results.

This paper is organized as follows. In section 2, a brief introduction to ring-averaged RDT and FMA is given, and the relationship between ring-averaged RDT and FMA result of DA is analyzed with the SSRF storage ring lattice [19] that was also used in our previous analysis [18]. Then in section 3, this relationship is further analyzed with the HALF storage ring lattice [20], where the variations in amplitude-dependent tune shifts (ADTSs) are constrained. In section 4, a comparison is made between the tracking-based optimization of minimizing frequency diffusion rates and the optimization of minimizing ring-averaged RDT. Finally, the conclusion is given in section 5.

## 2 Analysis of ring-averaged RDT and frequency diffusion

In the normal form analysis, the nonlinear part of the transfer map of a storage ring lattice can be expanded into a series of nonlinear driving terms using the resonance basis [11]. The terms driving resonances are referred to as RDTs, which vary along the longitudinal position [21, 22]. Reducing the ring-averaged RDTs is very effective in enlarging DA area [18, 23, 24]. The underlying physics is that higher-order nonlinear terms are controlled by reducing the lower-order ring-averaged RDT [18, 25]. The ring-averaged RDT for a single RDT is calculated as the mean value over the locations of nonlinear magnets [18]:

$$f_{jklm,\text{avg}} = \frac{1}{N} \sum_{i=1}^N |f_{jklm}(s_i)|, \quad (2.1)$$

where  $f_{jklm}(s_i)$  is the RDT calculated as in ref. [26], with  $s_i$  being the location of the  $i$ th nonlinear magnet, and  $N$  is the number of nonlinear magnets. Then the  $n$ th order ring-averaged RDT, denoted as  $f_{n,\text{avg}}$ , is calculated as [18]:

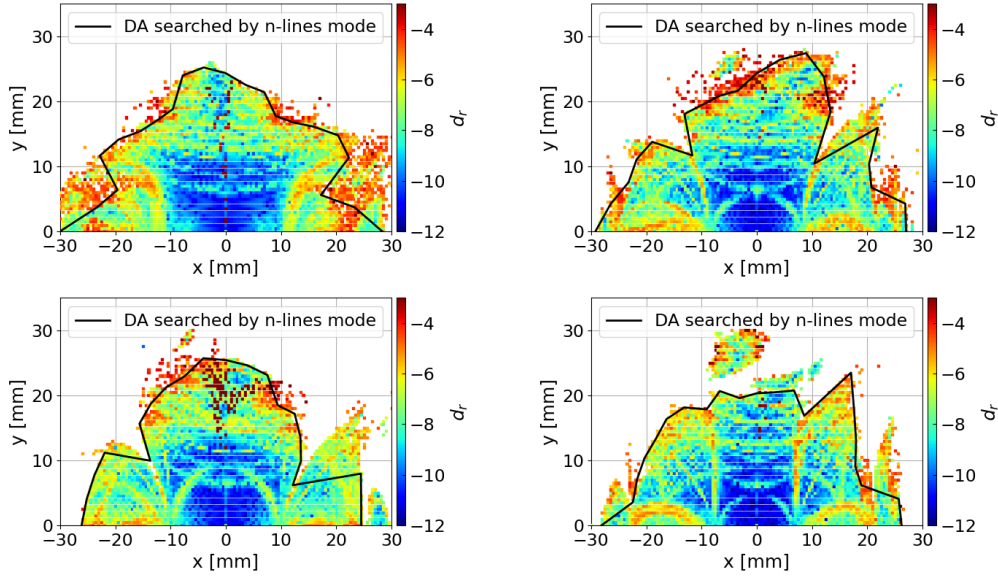
$$f_{n,\text{avg}} = \sqrt{\sum_{j+k+l+m=n} f_{jklm,\text{avg}}^2}. \quad (2.2)$$

Compared to the area of DA, FMA gives more information about DA. In FMA, the impact of resonances is revealed by analyzing the tunes of surviving particles and their frequency diffusion rates. In this paper, the frequency diffusion rate is calculated using *elegant* [27] as:

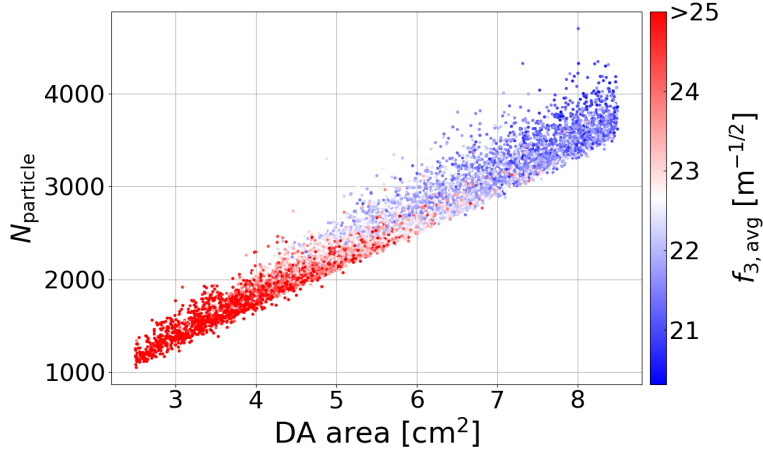
$$d_r = \log_{10} \left( \sqrt{\Delta v_x^2 + \Delta v_y^2 / N_{\text{turn}}} \right), \quad (2.3)$$

where  $\Delta v_x$  and  $\Delta v_y$  represent the frequency differences in the  $x$  and  $y$  directions between the first and second half of tracking turns, respectively, and  $N_{\text{turn}}$  is the total number of tracking turns. The NAFF algorithm [3] used in FMA converges quickly in tune calculations, typically requiring only a few hundred turns of tracking [8]. In this paper, particle tracking is performed in 4D, and  $N_{\text{turn}}$  is set to 516 for all lattice solutions.

In our previous work [18], although the strong effectiveness of reducing ring-averaged RDT in increasing DA area was demonstrated with a large number of nonlinear solutions of the SSRF storage ring lattice, it can still be observed that some solutions have similar DA areas but exhibit obvious differences in ring-averaged RDT. The DA areas were calculated with the *n*-lines search mode [28] in *elegant*. Here, also using *elegant*, a detailed grid search was performed for calculating the DAs and frequency diffusion rates of the SSRF nonlinear solutions from ref. [18], which were primarily



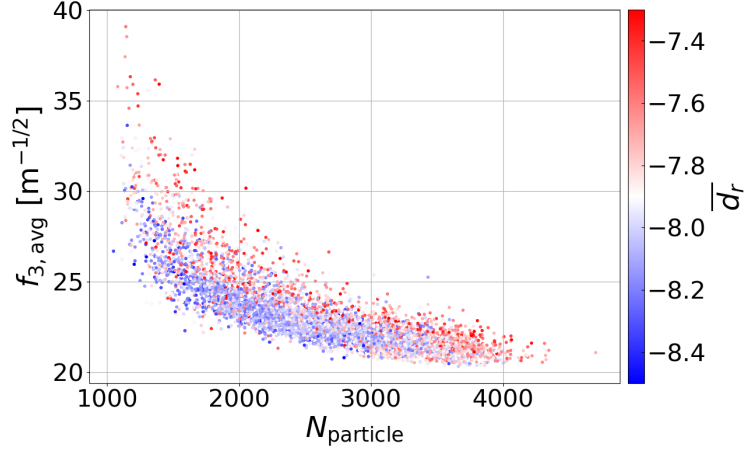
**Figure 1.** Comparison of DA results obtained by n-lines mode and grid mode for four nonlinear solutions of the SSRF storage ring lattice.



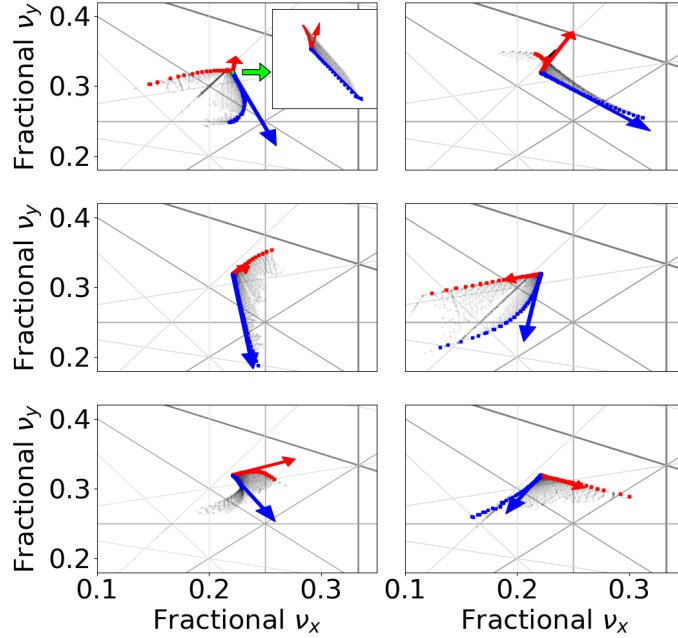
**Figure 2.** Correlation among the DA area obtained with the n-lines mode, the number of surviving particles ( $N_{\text{particle}}$ ) obtained with the grid mode and the third-order ring-averaged RDT ( $f_{3,\text{avg}}$ ) for the SSRF lattice.

optimized using genetic algorithm to have different DAs for better analysis. Figure 1 compares the DA results of the two search modes for four nonlinear solutions. It can be seen that some surviving particles located on islands or peninsulas separated by resonance lines were not found by the n-lines mode. Although the DA excluding islands is of primary interest, including the islands provides more complete information about nonlinear dynamics.

Figure 2 presents the correlation among the DA area obtained with the n-lines mode, the number of surviving particles obtained with the grid mode (denoted as  $N_{\text{particle}}$ ) and the third-order ring-averaged RDT  $f_{3,\text{avg}}$ . It is seen that, generally, for solutions with the same n-lines DA area, those with smaller  $f_{3,\text{avg}}$  have larger  $N_{\text{particle}}$ . This indicates that  $N_{\text{particle}}$  has a stronger correlation with ring-averaged RDT. The Spearman correlation coefficient [29] between  $f_{3,\text{avg}}$  and  $N_{\text{particle}}$  is -0.874 (95% confidence



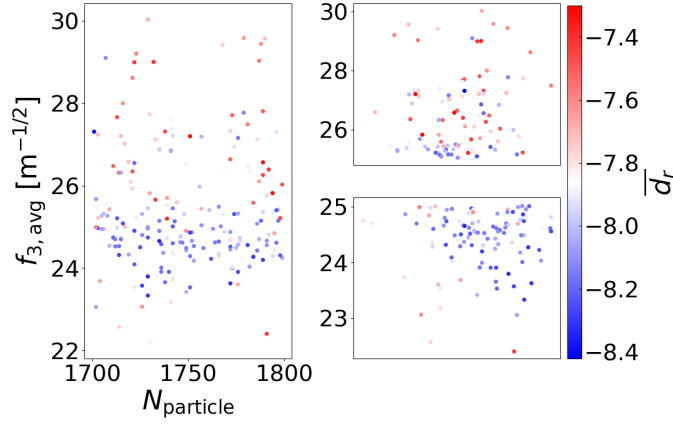
**Figure 3.** Relationship between the number of surviving particles, the average frequency diffusion rate and the third-order ring-averaged RDT for the SSRF lattice.



**Figure 4.** Frequency maps and ADTSs for six SSRF nonlinear solutions with DAs around  $6 \text{ cm}^2$ . Dots are tracked tunes, with red ones showing the tune shifts with horizontal amplitude and blue ones for vertical amplitude. Arrows show the corresponding first order ADTSs.

interval:  $-0.879$  to  $-0.869$ ), while the coefficient between  $f_{3,\text{avg}}$  and n-lines DA area is  $-0.841$  (95% confidence interval:  $-0.847$  to  $-0.835$ ).

Next, we further consider the frequency diffusion rate. Figure 3 shows the relationship between  $f_{3,\text{avg}}$ ,  $N_{\text{particle}}$  and the average frequency diffusion rate of surviving particles (denoted as  $\overline{d_r}$ ). We see that for solutions with the similar  $N_{\text{particle}}$ , those with smaller  $f_{3,\text{avg}}$  roughly exhibit lower  $\overline{d_r}$ . However, it is not rather obvious due to significant variations in both the magnitudes and directions of ADTSs for these solutions, which causes their DAs to be affected by different resonance lines. Figure 4 shows the



**Figure 5.** Comparison of two solution subsets with  $N_{particle}$  in the range of 1700 to 1800. The two subsets have the same number of solutions. The upper subset has a higher average  $\overline{d_r}$ , and the lower subset has a lower average  $\overline{d_r}$ .

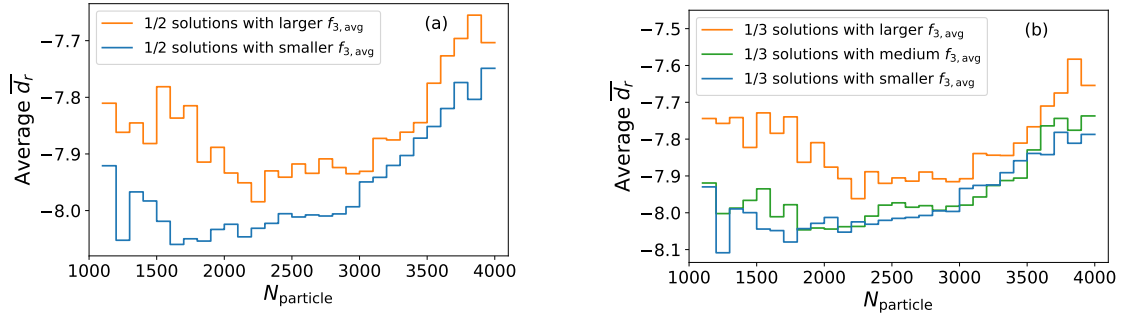
frequency maps of several nonlinear solutions with DA areas around  $6 \text{ cm}^2$ . Their first order ADTSs (red and blue arrows) differ from each other, and due to the higher-order effect, the tracked ADTSs (red and blue dots) deviate from the first order ADTSs to varying degrees, exhibiting more complex changes.

To reduce the impact of differences in ADTSs, we do not compare  $\overline{d_r}$  of individual solutions, but compare the average  $\overline{d_r}$  of solution subsets. In figure 5, the solutions with  $N_{particle}$  in the range of 1700 to 1800 are divided into two subsets with an equal number of solutions. For the subset with larger  $f_{3,avg}$ , the mean value of  $\overline{d_r}$  is  $-7.814$ ; for the subset with smaller  $f_{3,avg}$ , it is  $-8.048$ . Therefore, it is clear that on average, reducing ring-averaged RDT is beneficial for lowering  $\overline{d_r}$ .

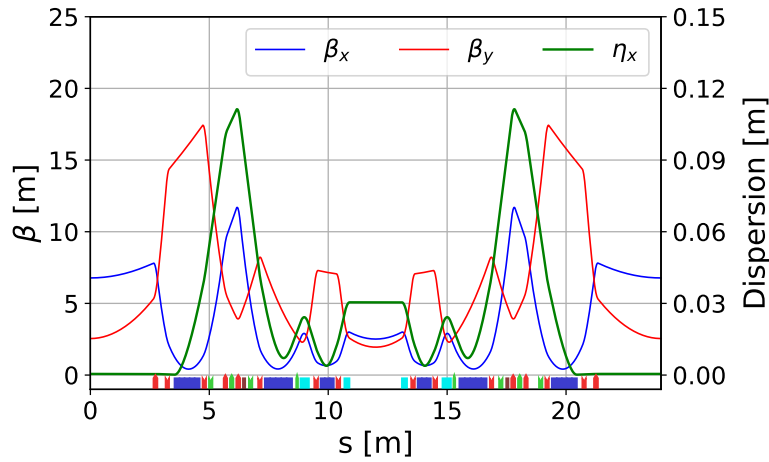
The same process was applied to the solutions with  $N_{particle}$  in a wider range of 1100 to 4000. Solutions outside this range, as shown in figure 3, are too sparse to reveal meaningful statistics and are therefore excluded from consideration. Within this wider range, the solutions in each subinterval, with a width of 100, were divided into two equally sized subsets based on their  $f_{3,avg}$  values, as done in figure 5, and the average  $\overline{d_r}$  for each subset was calculated. The subinterval width was chosen by considering two factors: maintaining comparable DA sizes while ensuring a sufficient number of solutions within each subinterval for meaningful statistical analysis. The calculated results are shown in figure 6(a). The ensemble-averaged tendency is that, for comparable  $N_{particle}$ , reducing  $f_{3,avg}$  is associated with a lower average  $\overline{d_r}$ . If each subinterval is divided into more subsets, the relation between  $f_{3,avg}$  and average  $\overline{d_r}$  will not be well maintained. As shown in figure 6(b), where each subinterval has three subsets, for the subsets with smaller and medium  $f_{3,avg}$ , the two curves are very close and intersect. This is due to the impact caused by differences in ADTSs, as shown in figure 4. Next, we will constrain ADTSs for analysis.

### 3 Further analysis with constrained ADTS

In this section, the Hefei Advanced Light Facility (HALF) storage ring lattice is used for further analysis. To generate different solutions with roughly close ADTS values, we found that a highly effective way is to introduce different levels of beta-beating into the lattice while keeping the strengths

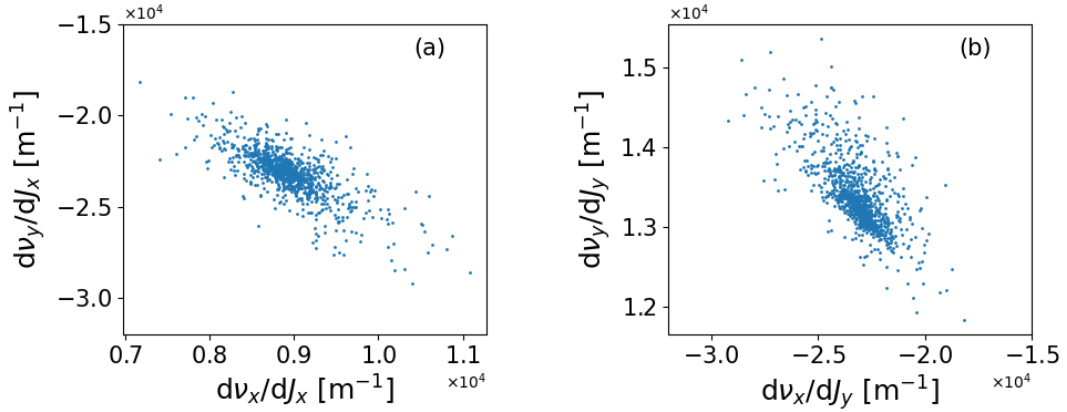


**Figure 6.** Comparison of average  $\overline{d_r}$  for different kinds of solution subsets. For each subinterval with a width of 100 in the  $N_{\text{particle}}$  range of 1100 to 4000, (a) the solutions are divided into two equal-sized subsets based on the values of  $f_{3,\text{avg}}$ ; (b) the solutions are divided into three equal-sized subsets.



**Figure 7.** Linear optical functions of a HALF storage ring lattice cell.

of nonlinear magnets unchanged. In this case, the corrected chromaticities of these solutions are essentially identical. The HALF storage ring consists of 20 modified hybrid six-bend-achromat (H6BA) lattice cells, which has a natural emittance of 86 pm [20, 30]. The HALF H6BA lattice adopts the  $-I$  cancellation scheme. The linear optical functions of a lattice cell are shown in figure 7. By introducing quadrupole strength errors, 1000 lattice solutions were generated, which have different levels of beta-beating, with a maximum value of about 20%. For these solutions, their working points were corrected to the same values, and the nonlinear magnet strengths were not changed. It was found that the ADTSs of these solutions do not change significantly as shown in figure 8. The signs of ADTSs for different lattice solutions are consistent, and the differences in ADTS values for most lattice solutions are within 20%. Thus, for these solutions, their ADTSs are constrained, and due to differences in beta-beating, their RDTs are different. Next, we will analyze these lattice solutions. The ring-averaged RDT and the betatron tunes affect the DA in different ways: the RDT quantifies the strength of the nonlinear perturbation, whereas the betatron tunes determine which resonances affect the DA. The focus of this paper is to isolate the relationship between RDT and DA. The following subsections consider two representative fixed working points separately, one on and one away from the resonance  $2\nu_x - 2\nu_y$ .



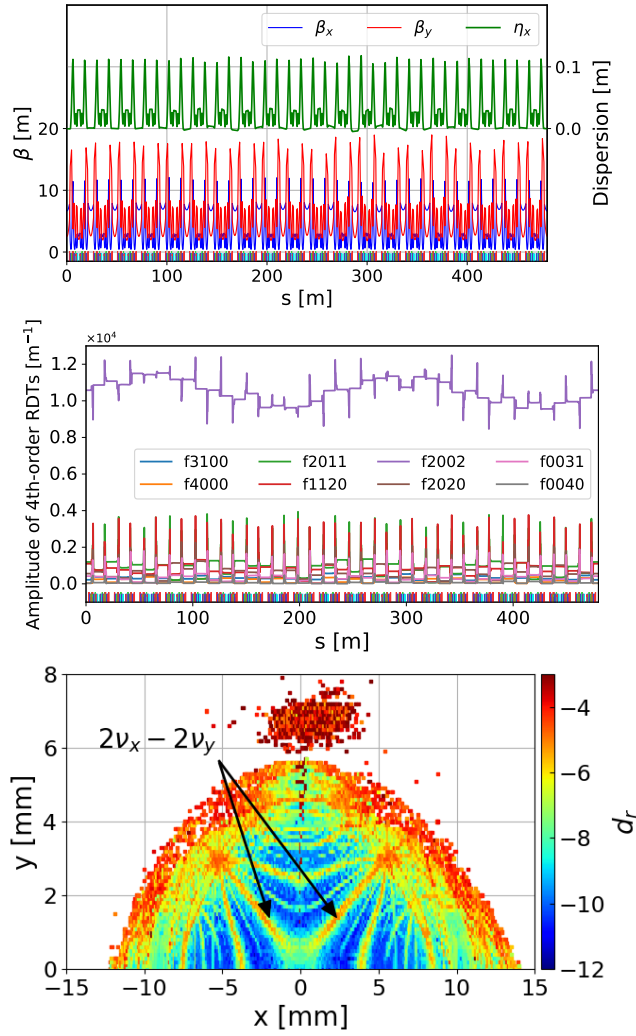
**Figure 8.** First order ADTSs of the HALF lattice solutions with different quadrupole strength errors: (a) tune shifts with horizontal amplitude, (b) tune shifts with vertical amplitude.

### 3.1 On-resonance working point

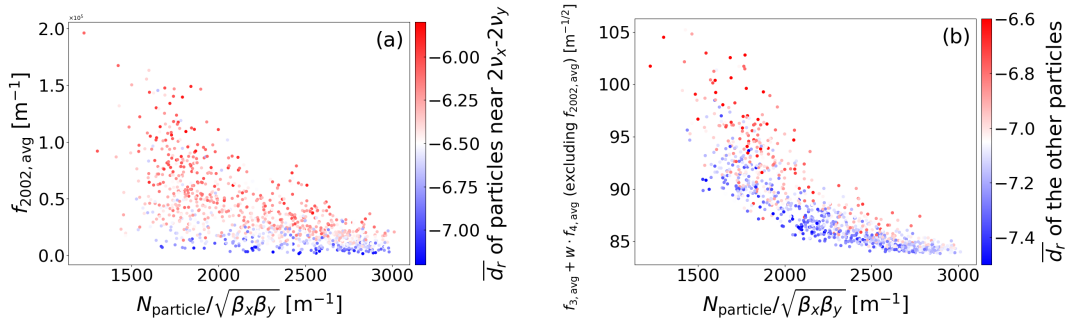
The working points of the lattice solutions have the same values of (48.19, 17.19), which is on the difference resonance. The frequency maps of the DAs of these solutions are analyzed using *elegant*. Figure 9 shows the linear optics, longitudinal variation of 4th-order RDTs and DA for an error solution. The error-induced  $\beta$ -beating breaks the symmetry of RDT variation, and it also worsens the nonlinear cancellation, resulting in an increase in RDTs. Among these RDTs,  $f_{2002}$  becomes significantly large, as the working point is on the corresponding resonance line  $2\nu_x - 2\nu_y$ . The errors excite resonances, leading to higher diffusion rates near the resonance lines and unstable particle motion.

Since the RDT  $f_{2002}$  is significantly larger than the other RDTs, it and its corresponding resonance  $2\nu_x - 2\nu_y$  are considered separately from the others. For each lattice solution, we calculated the  $\overline{d_r}$  of the particles near the resonance line ( $|2\nu_x - 2\nu_y| < 0.01$ ) and the  $\overline{d_r}$  of the other particles. Due to that the  $\beta$  function values at the middle of the straight section where DA is tracked are different for these solutions with errors, their  $N_{\text{particle}}$  need to be normalized by the  $\beta$  functions for better comparison. Similar to the normalized DA area, the normalized  $N_{\text{particle}}$  is calculated as  $N_{\text{particle}}/\sqrt{\beta_x\beta_y}$ . Figure 10 shows the relationship between normalized  $N_{\text{particle}}$ , ring-averaged RDT and  $\overline{d_r}$ . From figure 10(a), it is seen that reducing  $f_{2002,\text{avg}}$  effectively reduces the  $\overline{d_r}$  of the particles near the resonance  $2\nu_x - 2\nu_y$ , indicating that this resonance is effectively suppressed by minimizing  $f_{2002,\text{avg}}$ . In addition, we see that many solutions with small  $f_{2002,\text{avg}}$  do not have large normalized  $N_{\text{particle}}$ , which means that the normalized  $N_{\text{particle}}$  is mainly determined by the other RDTs as shown in figure 10(b). In figure 10(b), we also see that reducing the ring-averaged RDT of the other terms also reduces the  $\overline{d_r}$  of the other particles. Therefore, in both plots of figure 10, there is a stronger relationship between ring-averaged RDT and  $\overline{d_r}$  compared to the case of figure 3 where ADTSs are not constrained.

In our previous work in ref. [18], a 6BA lattice was studied, where the longitudinal variation of the resonance  $2\nu_x + 2\nu_y$  driving term was particularly large. Its DA area was primarily influenced by this RDT. This forms an interesting contrast with the situation here. The difference between these two situations lies in the fact that the sum resonance  $2\nu_x + 2\nu_y$  allowing for unlimited growth of amplitudes can lead to a loss of particles, while the difference resonance  $2\nu_x - 2\nu_y$  does not cause the loss but leads to an exchange of horizontal and vertical amplitudes.



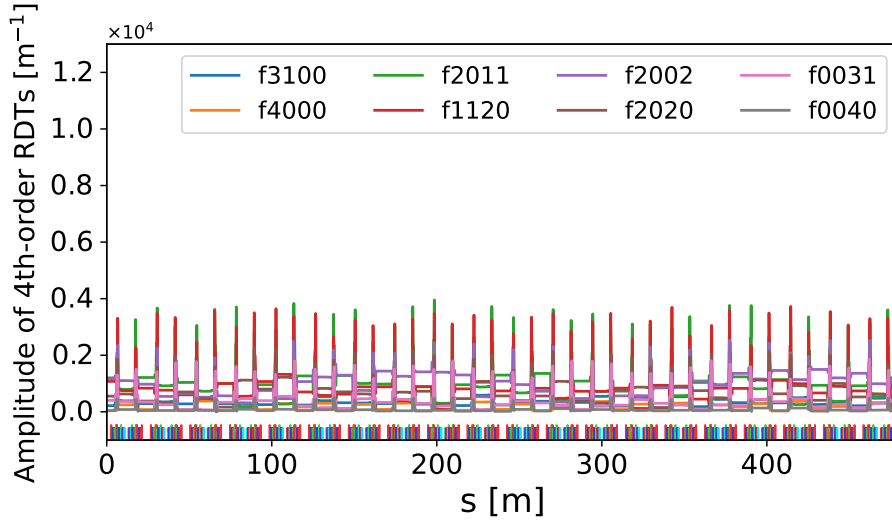
**Figure 9.** Linear optics (upper plot), longitudinal variation of 4th-order RDTs (middle plot) and DA (lower plot) of an error lattice of HALF. The working point is on the difference resonance, and the errors significantly increase the ring-averaged RDT of the resonance  $2\nu_x - 2\nu_y$ .



**Figure 10.** (a) Relationship between normalized  $N_{\text{particle}}$ , ring-averaged RDT  $f_{2002,\text{avg}}$  and  $\overline{d_r}$  of the particles with tunes satisfying  $|2\nu_x - 2\nu_y| < 0.01$ ; (b) relationship between normalized  $N_{\text{particle}}$ , the ring-averaged RDT of the other terms and  $\overline{d_r}$  of the other particles. In the calculation of ring-averaged RDT for plot (b), the weight coefficient  $w$  is set to  $1/250 \text{ m}^{1/2}$  so that the values of the third- and fourth-order terms are comparable.

### 3.2 Off-resonance working point

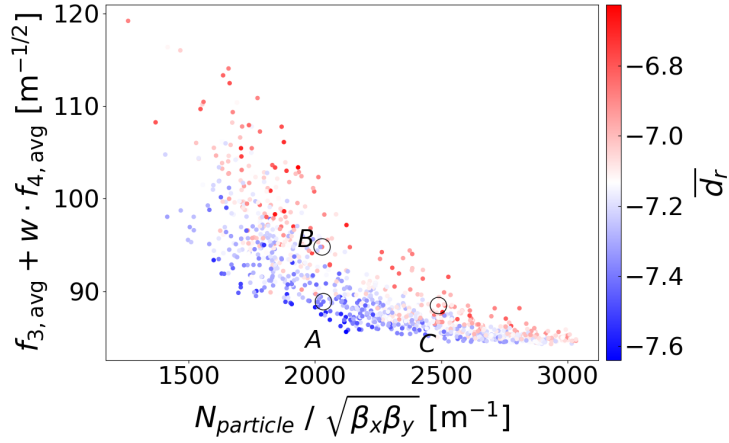
Next we study the case where the working point is not on the difference resonance, and in this case, the longitudinal variation of  $f_{2002}$  along the ring becomes smaller. Compared to the previous case of on-resonance, the distance between the working point and the resonance line  $2\nu_x - 2\nu_y$  is increased to 0.02 by slightly changing the strengths of quadrupoles. The 1000 error lattice solutions have the same off-resonance working point. Figure 11 shows the longitudinal variation of 4th-order RDTs of the error solution shown in figure 9 with the adjusted working point. We see that compared to figure 9,  $f_{2002}$  is significantly reduced, while other RDT variations remain almost unchanged.



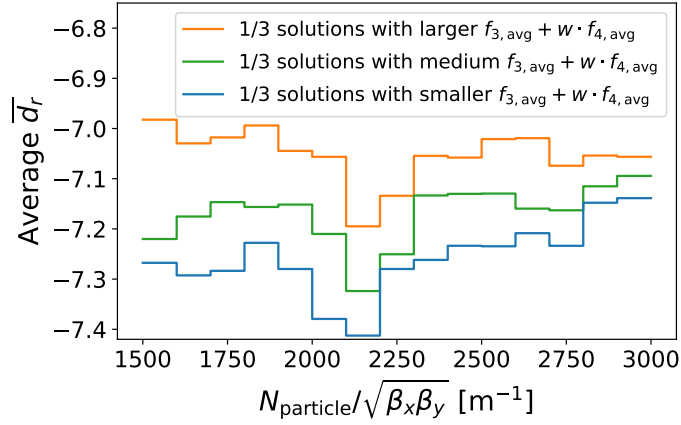
**Figure 11.** Longitudinal variation of fourth-order RDTs of the error lattice shown in figure 9 but with an off-resonance working point.

Figure 12 shows the relationship between normalized  $N_{\text{particle}}$ , ring-averaged RDT and  $\overline{d_r}$  of the error solutions with the adjusted working point. Due to that  $f_{2002}$  along the ring is smaller than before and becomes comparable to the other 4th-order RDTs, all the RDTs are considered together in the analysis of the figure. Like figure 6(b), the solutions in figure 12 with normalized  $N_{\text{particle}}$  in the range of 1500 to 3000 were split into subintervals with a width of 100, and each subinterval was divided into three equally sized subsets based on their ring-averaged RDT values. The average  $\overline{d_r}$  for each subset was calculated, shown in figure 13. It is seen that, different from figure 6(b), here the three step-like curves do not intersect, indicating that reducing ring-averaged RDT has a stronger relationship with lowering frequency diffusion rates when ADTSs are constrained.

Three solutions marked in figure 12, denoted as A, B and C, were selected for better illustration. The ring-averaged RDT, normalized  $N_{\text{particle}}$  and  $\overline{d_r}$  for the three solutions are listed in table 1, and their FMA results are shown in figure 14. Their  $\sqrt{\beta_x \beta_y}$  values at the DA tracking location are almost the same, making the comparison of the DAs in figure 14 more straightforward. We see that A and B have similar DA sizes, but A, with smaller ring-averaged RDT, has weaker resonances and thus more surviving particles with stable motion in its DA. For A and C with similar ring-averaged RDT, C has a larger DA but more surviving particles have high diffusion rates. These three solutions clearly illustrate the importance of reducing ring-averaged RDT in lowering frequency diffusion rates.



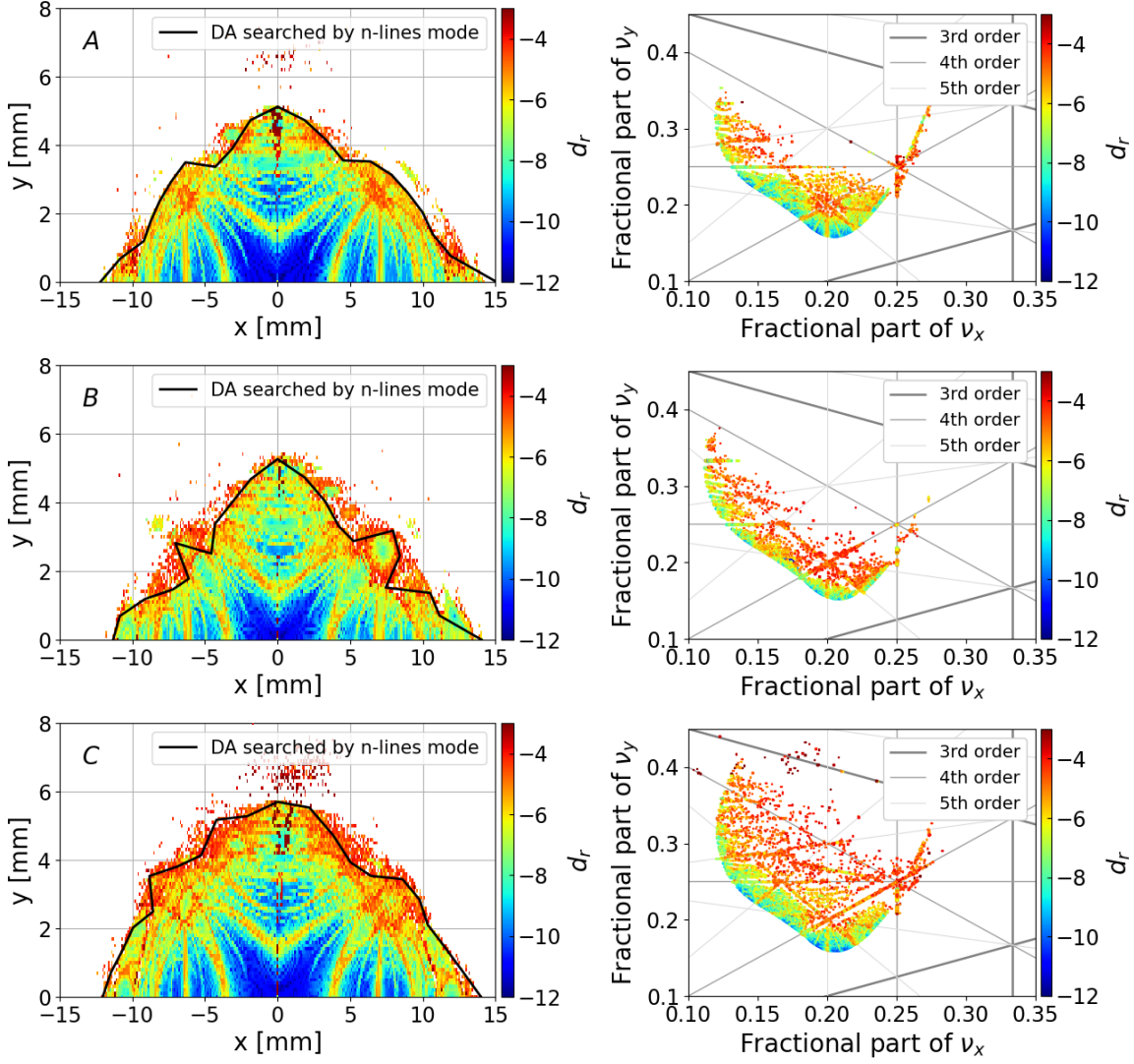
**Figure 12.** Relationship between normalized  $N_{\text{particle}}$ , ring-averaged RDT and  $\overline{d_r}$  of all surviving particles. In the calculation of ring-averaged RDT, the weight coefficient  $w$  is also set to  $1/250 \text{ m}^{1/2}$ . Three solutions marked with circles, denoted as A, B and C, are selected for the comparison shown in table 1 and figure 14.



**Figure 13.** Comparison of the average  $\overline{d_r}$  of HALF error lattice solution subsets with larger, medium and smaller ring-averaged RDT values for each subinterval with a width of 100 in the normalized  $N_{\text{particle}}$  range of 1500 to 3000.

**Table 1.** Ring-averaged RDT, normalized  $N_{\text{particle}}$  and  $\overline{d_r}$  of the three solutions marked in figure 12. Their DAs and frequency maps are shown in figure 14.

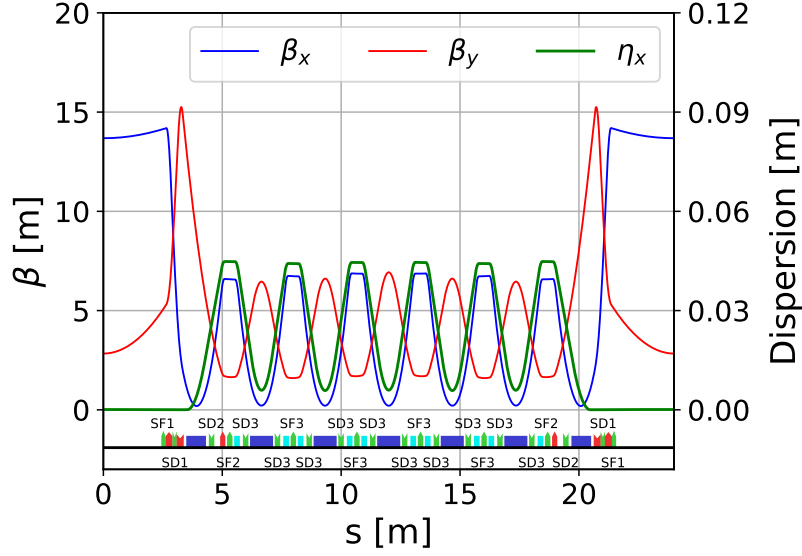
|                                                                | A     | B     | C     |
|----------------------------------------------------------------|-------|-------|-------|
| $f_{3,\text{avg}} [\text{m}^{-1/2}]$                           | 74.61 | 75.75 | 74.29 |
| $f_{4,\text{avg}} [\text{m}^{-1}]$                             | 3566  | 4760  | 3541  |
| $N_{\text{particle}} / \sqrt{\beta_x \beta_y} [\text{m}^{-1}]$ | 2033  | 2028  | 2489  |
| $\overline{d_r}$                                               | -7.41 | -6.88 | -6.90 |



**Figure 14.** FMA results of the DAs for the three solutions marked in figure 12. The ring-averaged RDT, normalized  $N_{\text{particle}}$  and  $\overline{d_r}$  for the three solutions are listed in table 1. The solutions A and C have similar ring-averaged RDT, while B has larger ring-averaged RDT.

#### 4 DA optimization and comparison

In refs. [8, 10], minimizing the sum of the frequency diffusion rates of particles was used as an objective in the DA optimization, leading to a robust DA. Since reducing ring-averaged RDT effectively enlarges DA and lowers frequency diffusion rates, it can also serve as an objective for optimizing DA. In this section, a seven-bend-achromat (7BA) lattice [31] is used as an example for DA optimization, which was designed for the HALF study but not used as the baseline lattice. Its magnet layout and linear optics of a lattice cell is shown in figure 15. The storage ring consists of 14 cells, and have a natural emittance of 68.4 pm at 2.2 GeV. There are six families of sextupoles in the storage ring, which are also shown in figure 15. In the DA optimization, the strengths of sextupole families SD1, SF1, SD2 and SF2 were set as optimization variables, while SD3 and SF3 were used to correct the horizontal and vertical chromaticities to (2, 2).



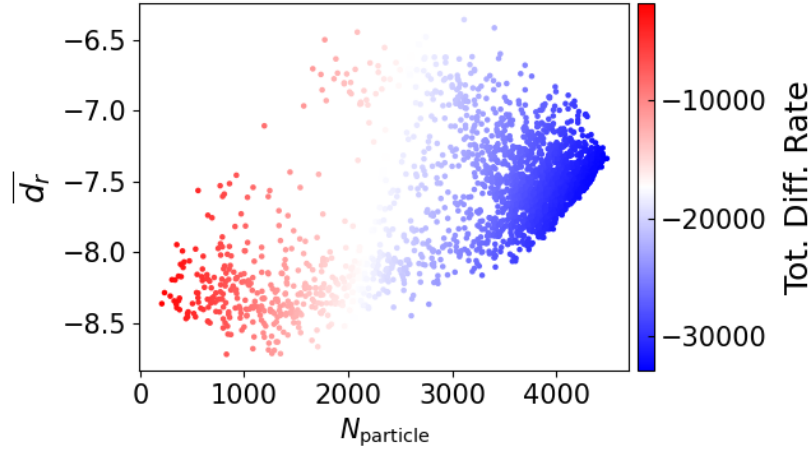
**Figure 15.** Linear optical functions and magnet layout of a 7BA lattice cell. Six families of sextupoles are listed in the magnet layout.

Three optimization schemes with different objectives were conducted for comparison. The first scheme follows the optimization in ref. [10], using minimizing the total frequency diffusion rate as the objective. The total diffusion rate is calculated by the summation of all the particle diffusion rates. The tracking was done in the  $x$ - $y$  space of  $(-18, 15) \text{ mm} \times (0, 6) \text{ mm}$ . Reducing the total diffusion rate increases  $N_{\text{particle}}$  and lowers  $\overline{d_r}$ . The second and third schemes minimize ring-averaged RDT, but use different quantitative representations of ring-averaged RDT. In the second scheme, the 3rd-order average RDT  $f_{3,\text{avg}}$  was minimized. Recently, in the DA optimization of the SOLEIL II storage ring lattice, we found that reducing the root mean square (RMS) RDTs over the locations of nonlinear magnets led to better solutions. The 3rd-order RMS RDTs, denoted as  $f_{3,\text{rms}}$ , is calculated as

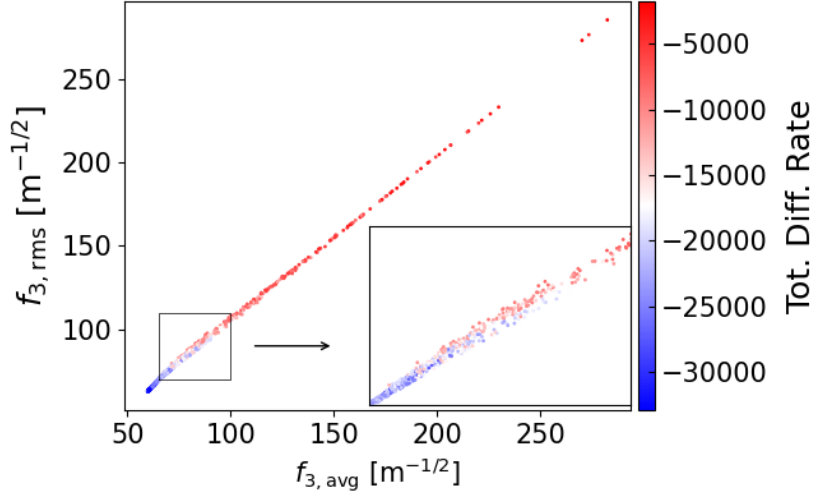
$$f_{3,\text{rms}} = \sqrt{\sum_{j+k+l+m=3} f_{jklm,\text{rms}}^2}, \quad \text{with } f_{jklm,\text{rms}} = \sqrt{\frac{1}{N} \sum_{i=1}^N |f_{jklm}(s_i)|^2}, \quad (4.1)$$

which was minimized in the third scheme. Fourth-order RDTs were not considered here since no octupoles were employed, meaning that reducing 3rd-order ring-averaged RDT can effectively reduce 4th-order ring-averaged RDT [18, 25]. In addition, the second-order chromaticities were constrained in the three optimization schemes as:  $|\xi_x^{(2)}| < 100$ ,  $|\xi_y^{(2)}| < 100$ .

The differential evolution algorithm [32, 33] implemented in pymoo [34] was used for the three schemes. It was iterated for 50 generations with a population size of 50 for each scheme. For the first scheme,  $N_{\text{particle}}$  and  $\overline{d_r}$  of all solutions generated in the optimization were also calculated, as shown in figure 16. It is seen that as the total diffusion rate was reduced in the optimization, the primary effect is an increase in  $N_{\text{particle}}$ , and for the same  $N_{\text{particle}}$ ,  $\overline{d_r}$  was reduced. We also calculated  $f_{3,\text{avg}}$  and  $f_{3,\text{rms}}$  for all generated solutions in the first scheme. The Spearman correlation coefficients of the total diffusion rate with  $f_{3,\text{avg}}$  and  $f_{3,\text{rms}}$  are both as high as 0.91. As shown in figure 17,  $f_{3,\text{rms}}$  and  $f_{3,\text{avg}}$  are strongly correlated. However, we still see that reducing  $f_{3,\text{rms}}$  is more effective in lowering the total diffusion rate, as solutions with similar  $f_{3,\text{avg}}$  but smaller  $f_{3,\text{rms}}$  exhibit lower total



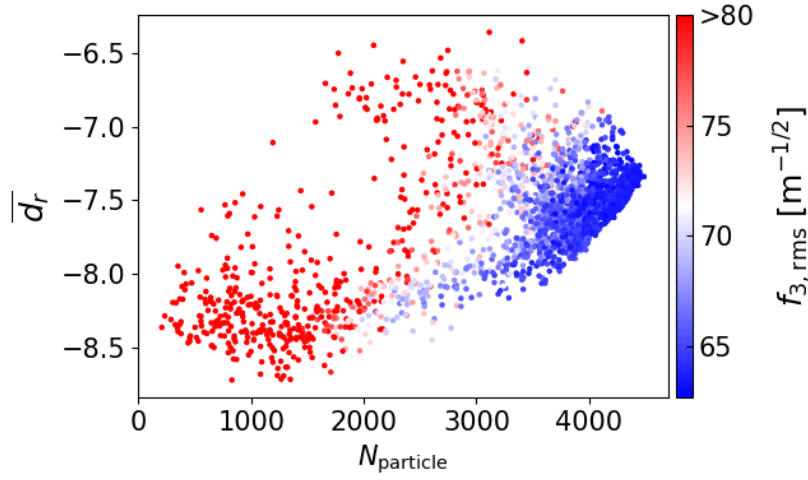
**Figure 16.** Change of  $N_{\text{particle}}$  and  $\overline{d_r}$  with the total diffusion rate for all nonlinear solutions generated in the first scheme, where the total diffusion rate is minimized in the optimization.



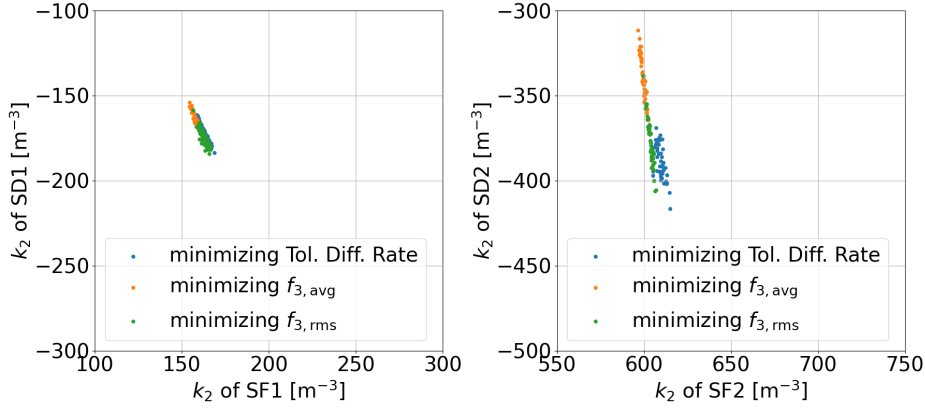
**Figure 17.** Correlation among  $f_{3,\text{avg}}$ ,  $f_{3,\text{rms}}$  and the total diffusion rate for all nonlinear solutions generated in the first scheme. Solutions with similar  $f_{3,\text{avg}}$  but smaller  $f_{3,\text{rms}}$  generally exhibit lower total diffusion rate.

diffusion rate. Figure 18 shows the relationship between  $N_{\text{particle}}$ ,  $\overline{d_r}$  and  $f_{3,\text{rms}}$  for these solutions. Compared with figure 16, we observe that the region with small  $f_{3,\text{rms}}$  is consistent with the region featuring low total diffusion rate. For solutions with comparable  $N_{\text{particle}}$ , those with smaller  $f_{3,\text{rms}}$  exhibit lower  $\overline{d_r}$ ; conversely, for solutions with similar  $\overline{d_r}$ , smaller  $f_{3,\text{rms}}$  corresponds to larger  $N_{\text{particle}}$ . Therefore, reducing  $f_{3,\text{rms}}$  not only enlarges DA but also stabilizes particle motion within it. It is important to note that there exists an inherent trade-off between  $N_{\text{particle}}$  and  $\overline{d_r}$ . Both reducing the total diffusion rate and reducing  $f_{3,\text{rms}}$  aim at finding solution regions where  $N_{\text{particle}}$  is large and  $\overline{d_r}$  is relatively low, as shown by the blue dots in figures 16 and 18.

Figure 19 shows the distribution of the solutions of the last generation in the variable space for the three schemes. We see that the regions of optimized solutions found by minimizing ring-averaged RDT, particularly minimizing  $f_{3,\text{rms}}$ , are close to the solution regions optimized by minimizing the

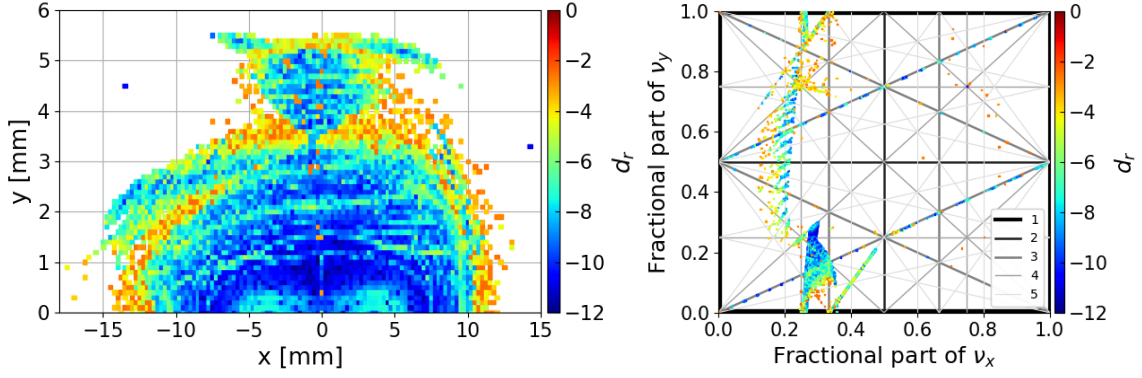


**Figure 18.** Relationship between  $N_{\text{particle}}$ ,  $\overline{d_r}$  and  $f_{3,\text{rms}}$  for all nonlinear solutions generated in the first scheme.

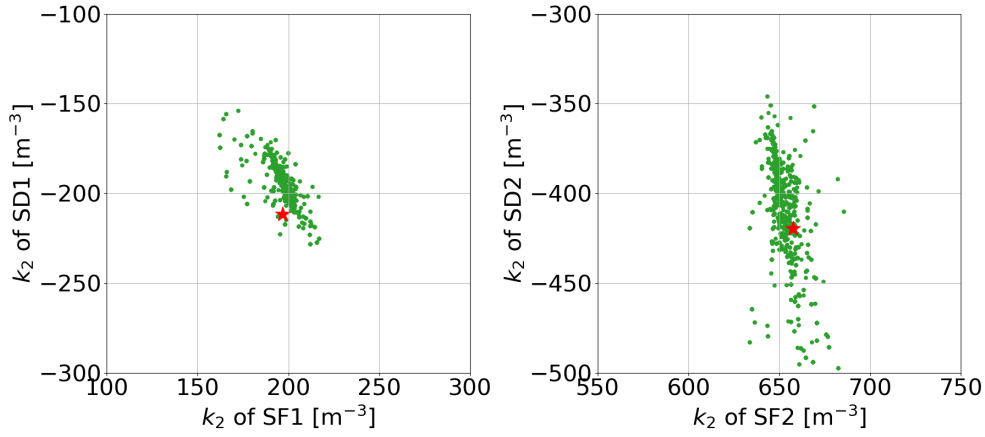


**Figure 19.** Distribution of the solutions of the last generation in the variable space for three DA optimization schemes with different objectives.

total diffusion rate. While  $f_{3,\text{avg}}$  and  $f_{3,\text{rms}}$  exhibit only marginal differences in figure 17, figure 19 focuses on the optimal solutions of different schemes, where such minor differences become distinct. Therefore, from the perspective of comparing these three schemes, it also indicates that reducing ring-averaged RDT very effectively lowers total diffusion rate and reducing  $f_{3,\text{rms}}$  is more effective. It is worth noting that the computational effort required for calculating ring-averaged RDT is much smaller than that for calculating total diffusion rate based on particle tracking. The optimization of minimizing total diffusion rate was performed in parallel using 24 CPUs on a desktop computer for approximately nine days, whereas minimizing ring-averaged RDT took only about one minute in parallel using 4 CPUs on a laptop. Therefore, minimizing ring-averaged RDT is a very effective approach for DA optimization to quickly find solution regions with high-quality DAs. Figure 20 presents the FMA of DA for one solution in the last generation of the third scheme with  $f_{3,\text{rms}}$  being minimized. The values of  $N_{\text{particle}}$ ,  $\overline{d_r}$  and the total diffusion rate of this solution were calculated to be 4206, -7.4 and -4.8, respectively, which are close to those of the optimal solution with minimum total diffusion rate in the first scheme as shown in figure 16. The FMA reveals that the horizontal



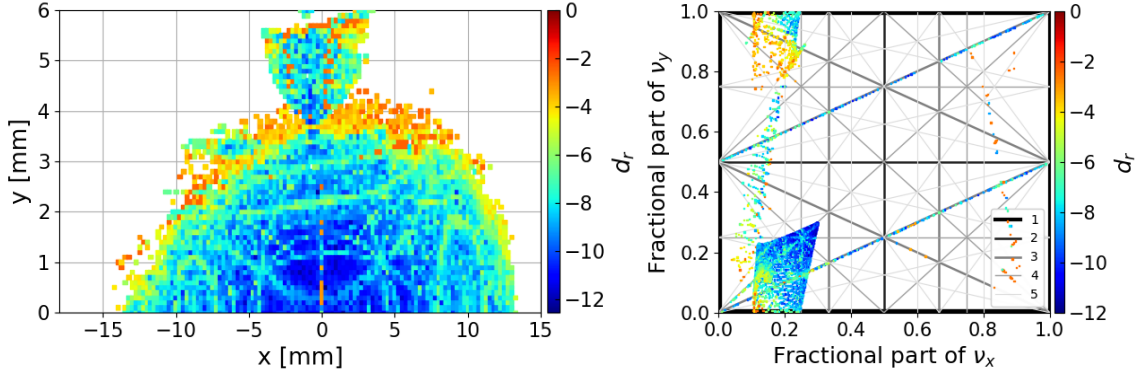
**Figure 20.** FMA of DA for one solution in the last generation of the third scheme, where  $f_{3,\text{rms}}$  was minimized. The value of  $f_{3,\text{rms}}$  for this solution is  $62.9 \text{ m}^{-1/2}$ .



**Figure 21.** Distribution in the variable space for good solutions obtained by minimizing  $f_{3,\text{rms}}$  with an ADTS constraint. The values of  $f_{3,\text{rms}}$  for these solutions are less than  $67 \text{ m}^{-1/2}$ . The nonlinear solution in ref. [31] is marked with a red star.

DA is affected by higher-order ADTS effect, which causes the frequency map to be folded, and thus the horizontal DA is limited by some resonances.

To further enlarge the horizontal DA, which is more important than the vertical DA for this lattice, an additional constraint on ADTS was introduced to ensure a significantly large negative  $\partial v_x / \partial J_x$ , preventing this folding. Specifically, when  $x = 10 \text{ mm}$ , the change of  $v_x$  due to the first-order ADTS  $\partial v_x / \partial J_x$  is:  $\Delta v_x < -0.2$ . Then, a new DA optimization aimed at minimizing  $f_{3,\text{rms}}$  with this ADTS constraint was conducted. The minimum  $f_{3,\text{rms}}$  obtained by the optimization is about  $64.9 \text{ m}^{-1/2}$ . Figure 21 shows the distribution in the variable space for all the solutions with  $f_{3,\text{rms}}$  less than  $67 \text{ m}^{-1/2}$  in the optimization. It can be seen that the solution region obtained is close to the solution in ref. [31]. Figure 22 shows the FMA of DA for one good solution in figure 21. It is seen that the horizontal DA is enlarged and particles near the edge of DA are more stable due to that the frequency map folding does not occur at small  $x$ . Compared to the solution with its DA shown in figure 20, this solution has a larger  $f_{3,\text{rms}}$  of  $66.7 \text{ m}^{-1/2}$  and a higher total diffusion rate of  $-4.7$ . However, its larger horizontal DA is more favorable for off-axis injection.



**Figure 22.** FMA of DA for one good solution in figure 21 with  $f_{3,\text{rms}}$  of  $66.7 \text{ m}^{-1/2}$ .

## 5 Conclusion

In this paper, the relationship between ring-averaged RDT and DA was further enhanced through FMA, which reveals the stability of particle motion and thus indicate the quality of DA. Using the FMA of DA, two quantities, the number of surviving particles  $N_{\text{particle}}$  and their average frequency diffusion rate  $\overline{d_r}$ , can be obtained, which were used to better study the relationship between DA and ring-averaged RDT. The FMA in this paper was performed with the grid search using *elegant*. Compared to the DA area calculated with the line search mode in *elegant*,  $N_{\text{particle}}$  has a stronger correlation with ring-averaged RDT. Then, the relationship between ring-averaged RDT and  $\overline{d_r}$  was analyzed for nonlinear solutions with similar  $N_{\text{particle}}$ . In the analysis, solutions with similar  $N_{\text{particle}}$  were divided into several subsets according to their ring-averaged RDT values. Subsets with smaller ring-averaged RDT values have lower average  $\overline{d_r}$ , which becomes more apparent when ADTSs are constrained, as the DAs of nonlinear solutions with different ADTSs are influenced by different resonances. This also holds true for a single resonance and its surrounding regions, as demonstrated by analyzing the HALF lattice with an on-resonance working point. Besides, for nonlinear solutions with similar ring-averaged RDT but different  $N_{\text{particle}}$ , those with larger  $N_{\text{particle}}$  tend to have higher  $\overline{d_r}$ . Therefore, reducing ring-averaged RDT generally increases  $N_{\text{particle}}$  and lowers  $\overline{d_r}$ , thereby improving DA.

Furthermore, a comparison was made among three DA optimization schemes with different objectives: one aimed at minimizing frequency diffusion rates based on particle tracking, while the other two aimed at minimizing ring-averaged RDT, represented by the average RDT  $f_{3,\text{avg}}$  and the RMS RDT  $f_{3,\text{rms}}$ , respectively. It was found that the regions of optimal solutions obtained by minimizing ring-averaged RDT, especially by minimizing  $f_{3,\text{rms}}$ , were close to those obtained by the tracking-based optimization. Moreover, the analysis of the nonlinear solutions from the tracked-based optimization also demonstrated that minimizing  $f_{3,\text{rms}}$  is more effective in lowering frequency diffusion rates than minimizing  $f_{3,\text{avg}}$ . In the optimization of minimizing ring-averaged RDT, by imposing an ADTS constraint to avoid some resonances, the horizontal DA was further enlarged and particles near the edge of DA became more stable.

So far, the relationship between ring-averaged RDT and DA has been better established. Reducing ring-averaged RDT not only enlarges DA but also makes particle motion within it more stable. Similar analysis can be expected to hold for off-momentum ring-averaged RDTs [35] and off-momentum DAs. Due to the low computational cost, minimizing ring-averaged RDT can be used to efficiently

optimize the nonlinear dynamics of lattices. Moreover, by optimizing the magnet layout and parameters with a focus on minimizing ring-averaged RDT, it is possible to explore new lattice structures with relatively good nonlinear performance.

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